MATH 579 Exam 5 Solutions

1. Calculate the number of compositions of 14 into an even number of even parts.

   Partitions of 14 into even parts are bijective with partitions of 7 into integer parts, by dividing each part by 2. We don't want all such though, we insist on an even number of parts, namely 2 or 4 or 6. Applying Cor. 5.3 thrice, the answer is $\binom{6}{1}+\binom{6}{3}+\binom{6}{5}=6+20+6=32$.

2. For all $n\in\mathbb{N}$, determine $S(n,n-2)$.

   There are two types of set partitions of $[n]$ into $n-2$ parts. First, there is the type that has one triple and $n-3$ singletons. There are $\binom{n}{3}$ such. Second, there is the type that has two doubles and $n-4$ singletons. If the doubles were different, there would be $\binom{n}{2}\binom{n-2}{2}$ such; however, they are not, so in fact there are $\frac{1}{2!}\binom{n}{2}\binom{n-2}{2}$ such. Putting it together, we get $\binom{n}{3}+\frac{1}{2!}\binom{n}{2}\binom{n-2}{2}$. Note that this works even for $n=1,2$, where everything is 0.

3. Calculate $S(8,3)$.

   Using the helpful but not necessary formula $S(n,2)=2^{n-1}-1$, together with Thm 5.8, we get $S(3,3)=1, S(4,3)=S(3,2)+3S(3,3)=(2^2-1)+3=6, S(5,3)=S(4,2)+3S(4,3)=(2^3-1)+3(6)=25, S(6,3)=S(5,2)+3S(5,3)=(2^4-1)+3(25)=90, S(7,3)=S(6,2)+3S(6,3)=(2^5-1)+3(90)=301, S(8,3)=S(7,2)+3S(7,3)=(2^6-1)+3(301)=966$.

4. Let a_n denote the number of compositions of n where each part is larger than 1. Find a formula relating a_n, a_{n-1}, a_{n-2}.

   We divide such compositions into two types: A: those that have first term equal to 2, B: those that have first term greater than 2. Type A are bijective with compositions counted by a_{n-2}, as seen by removing that first term. Type B are bijective with compositions counted by a_{n-1}, as seen by subtracting one from the first term. Hence $a_n=a_{n-1}+a_{n-2}$. Note that $a_2=1, a_3=1$, so in fact these are the Fibonacci numbers in disguise.

5. For all $l,m,n\in\mathbb{N}_0$, prove that $\sum_k\binom{n}{k}S(k,l)S(n-k,m)=S(n,l+m)\binom{l+m}{l}$.

   We count partitions of $[n]$ into l nonempty "red" parts, and m nonempty "blue" parts. One way to do this is to first partition $[n]$ into $l+m$ nonempty parts, and then paint l of them red (the rest are blue). The RHS counts this way. Another way is to first choose k elements that will be in a red part; we then partition them into nonempty parts in $S(k,l)$ ways. The remaining $n-k$ elements will be in a blue part; we partition them in $S(n-k,m)$ ways. The LHS counts this approach.

6. For every prime p, prove that $B(p)\equiv 2 \pmod{p}$. Equivalently, prove that p divides $B(p)-2$.

   Consider the function f on partitions of $[p]$ that acts by permuting the numbers within the parts as $1\to 2\to 3\to\cdots\to p\to 1$. For example, for $p=3$, f acts as $\{1,2\}\{3\}\to\{2,3\}\{1\}\to\{1,3\}\{2\}\to\{1,2\}\{3\}$. Call two partitions 'equivalent' if some number of applications of f will map one onto the other. f leaves exactly two partitions alone: $\{1\}\{2\}\cdots\{p\}$ and $\{1,2,\ldots,p\}$. All other partitions are equivalent to exactly p partitions [special case of Lagrange's theorem]; hence $B(p)$ is two plus some multiple of p.

   Note 1: Since the cycle of partitions that f induces all have the same number of parts, this also proves that $p|S(p,k)$, for p prime and $1<k<p$.
   Note 2: p must be prime for this to hold. For example, for $p=4$, the cycle $\{1,2\}\{3,4\}\to\{2,3\}\{1,4\}\to\{1,2\}\{3,4\}$ only has two partitions, not p. And indeed $B(4)=15$, which is not congruent to 2 modulo 4.
   Note 3: This result is a special case of Touchard's Congruence: $B_n+B_{n+1}\equiv B_{n+p} \pmod{p}$. This problem corresponds to $n=0$; the general result can be proved in a similar way.

Exam results: High score=88, Median score=70, Low score=53 (before any extra credit)